UDC 531.36 + 62 - 50

ON CONTROL AND STABILIZATION OF SYSTEMS WITH IGNORABLE COORDINATES

PMM Vol. 36, №6, 1972, pp. 966-976

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(Received July 3, 1972)

We consider the problem of control and stabilization with respect to position coordinates and momenta of a holonomic system with ignorable coordinates by means of controls applied to the system, in particular, only relative to the ignorable coordinates. The problem is solved on the basis of methods from stability theory [1] and control theory [2]. Examples are presented.

We consider a holonomic mechanical system whose generalized coordinates and momenta are denoted by q_i, p_i $(i = 1, \ldots, n)$. The equations of motion of the system are written in the form of canonic Hamiltonian equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + Q_i \qquad (i = 1, \ldots, n) \tag{1}$$

where $H = H(t, q_i, p_i)$ is the Hamiltonian function, $Q_i = Q_i(t, q_j, p_j)$ are generalized nonpotential forces. We assume that the functions $H(t, q_i, p_i)$ and $Q_i(t, q_i, p_i)$ do not depend explicitly on coordinates q_α, i.e. the identities

$$\frac{\partial H}{\partial q_\alpha} \equiv 0, \qquad \frac{\partial Q_i}{\partial q_\alpha} \equiv 0 \qquad (i = 1, \ldots, n;\ \alpha = k + 1, \ldots, n) \tag{2}$$

are fulfilled. We call such coordinates q_α ignorable coordinates; in case $Q_\alpha \neq 0$ they are sometimes called quasi-ignorable coordinates. The remaining coordinates q_j $(j = 1, \ldots, k)$ are called position coordinates.

When conditions (2) are fulfilled the study of the system's motion is reduced to the investigation of the $2k$ equations

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} + Q_j \qquad (j = 1, \ldots, k) \tag{3}$$

for the position coordinates and momenta and of the $n - k$ equations

$$dp_\alpha / dt = Q_\alpha \qquad (\alpha = k + 1, \ldots, n) \tag{4}$$

for the ignorable momenta, after the integration of which the ignorable coordinates are determined by quadratures [1]. Thus, when ignorable coordinates are present it is possible to disregard them and to investigate the system of Eqs. (3), (4) of order $n + k$ instead of the original system of Eqs. (1) of order $2n$.

For Eqs. (3), (4) we can pose general problems of control and stabilization of motion [2], whose solutions would serve for the original system (1) as the solutions of problems of control and stabilization of motion with respect to a part of the variables [3], namely, with respect to the position coordinates and to all momenta. Besides such a general statement of the problem of control and stabilization for system (3), (4) it also makes sense to pose the particular problem when the controls are applied to the mechanical system only relative to the ignorable coordinates. This signifies the control of the system by the motion of those of its parts which can perform cyclic motions. Thus, for example, when the system is under the action only of potential forces and all $Q_\alpha = 0$, by virtue of Eqs. (4) the ignorable momenta p_α remain constant throughout the whole motion and, by selecting their magnitudes in suitable fashion the reduced system (3) can be controlled and the stabilization of its motions effected. A yet greater possibility for system control arises when the momenta p_α do not remain constant but vary in accordance with a specific law in correspondence with Eqs. (4), thanks to the fact that the controls Q_α are applied to the system relative to its ignorable coordinates. Here the functions p_α, in Eqs. (3) of motion of the reduced system, play the role of controls whose determination depends upon this or the other statement of the problem of controlling the reduced system (3) and, by the same token, the original system (1).

Let us consider this problem in detail. Suppose that for some given generalized forces $Q_\alpha = Q_\alpha(t)$ $(\alpha = k+1, \ldots, n)$ and for all $Q_j = 0$ $(j = 1, \ldots, k)$ Eqs. (3), (4) admit of the particular solution

$$q_j = q_j(t), \qquad p_i = p_i(t) \tag{5}$$

satisfying the given initial conditions. We accept this solution as the unperturbed motion of the system. Let the values of the coordinates and momenta in the perturbed motion be

$$q_j = q_j(t) + \xi_j, \qquad p_i = p_i(t) + \eta_i$$

where ξ_j, η_i denote the deviations or variations of the variables q_j and p_i. We write the equations of perturbed motion of the reduced system in the form of the canonic equations

$$\frac{d\xi_j}{dt} = \frac{\partial H_1}{\partial \eta_j}, \quad \frac{d\eta_j}{dt} = -\frac{dH_1}{\partial \xi_j} \qquad (j = 1, \ldots, k) \tag{6}$$

where the function

$$H_1(t, \xi_j, \eta_i) = H(t, q_j(t) + \xi_j, p_i(t) + \eta_i) - H(t, q_j(t), p_i(t)) - \sum_{j=1}^{k}\left(\frac{\partial H}{\partial q_j}\xi_j + \frac{\partial H}{\partial p_j}\eta_j\right) - \sum_{\alpha=k+1}^{n}\frac{\partial H}{\partial p_\alpha}\eta_\alpha$$

The partial derivatives of function H occurring in this expression have been computed for solution (5). The controls η_α satisfy the differential equations

$$d\eta_\alpha/dt = P_\alpha, \qquad P_\alpha = Q_\alpha - Q_\alpha(t) \qquad (\alpha = k+1, \ldots, n) \tag{7}$$

The function $H_1(t, \xi_j, \eta_i)$ and its first partial derivatives with respect to ξ_j, η_i vanish for $\xi_j = \eta_i = 0$. We assume that the function $H_1(t, \xi_j, \eta_i)$ is a holomorphic function of the variables ξ_j, η_i. Its Maclaurin series expansion in the variations of the variables starts with the quadratic form

$$H_{12}(t, \xi_j, \eta_i) = \frac{1}{2}\sum_{i,j=1}^{k}\left(\frac{\partial^2 H}{\partial q_i \partial q_j}\xi_i\xi_j + 2\frac{\partial^2 H}{\partial q_i \partial p_j}\xi_i\eta_j + \frac{\partial^2 H}{\partial p_i \partial p_j}\eta_i\eta_j\right) + \frac{1}{2}\sum_{\alpha,\beta=k+1}^{n}\frac{\partial^2 H}{\partial p_\alpha \partial p_\beta}\eta_\alpha\eta_\beta + \sum_{i=1}^{k}\sum_{\alpha=k+1}^{n}\left(\frac{\partial^2 H}{\partial p_i \partial p_\alpha}\eta_i\eta_\alpha + \frac{\partial^2 H}{\partial q_i \partial p_\alpha}\xi_i\eta_\alpha\right)$$

whose coefficients are computed for solution (5).

The first approximation of Eqs. (6) of perturbed motion obviously has the form of Poincare's variational equations [1]. Introducing the notation

$$x_j = \xi_j, \qquad x_{k+j} = \eta_j, \qquad u_\alpha = \eta_{k+\alpha}$$

$$p_{ij} = \frac{\partial^2 H}{\partial p_i \partial q_j}, \quad p_{i,\,k+j} = \frac{\partial^2 H}{\partial p_i \partial p_j}, \quad p_{k+i,\,j} = -\frac{\partial^2 H}{\partial q_i \partial q_j}$$

$$p_{k+i,\,k+j} = -\frac{\partial^2 H}{\partial q_i \partial p_j}, \quad q_{i\alpha} = \frac{\partial^2 H}{\partial p_i \partial p_{k+\alpha}}, \quad q_{k+i,\,\alpha} = -\frac{\partial^2 H}{\partial q_i \partial p_{k+\alpha}}$$

$$(i, j = 1, \ldots, k;\ \alpha = 1, \ldots, n-k)$$

we write Eqs. (6) of perturbed motion in the form

$$\frac{dx_s}{dt} = \sum_{r=1}^{2k} p_{sr} x_r + \sum_{\alpha=1}^{n-k} q_{s\alpha} u_\alpha + X_s(t, x_r, u_\alpha) \qquad (s = 1, \ldots, 2k) \tag{8}$$

Here $X_s(t, x_r, u_\alpha)$ are terms which are nonlinear in the variables x_r, u_α whose structure is clear from the form of the right-hand sides of Eqs. (6). The Hamiltonian function H is a second-degree function in the generalized momenta p_i, therefore, the function H_1 is the same in the variables η_i. Obviously, the quantities u_α can enter into the nonlinear terms X_s of Eqs. (8) only in powers no higher than the first for $s = 1, \ldots, k$ and no higher than the second for $s = k + 1, \ldots, 2k$. These terms can be represented as a sum

$$X_s(t, x_r, u_\alpha) = X_s^{(0)} + X_s^{(1)} + X_s^{(2)} \tag{9}$$

where the superscripts denote the degree of homogeneity of the form relative to the variables $u_\alpha = \eta_{k+\alpha}$. Obviously, $X_s^{(0)} = X_s(t, x_r, 0)$; the form $X_s^{(2)} \equiv 0$ for $s = 1, \ldots, k$. We assume further that the right-hand sides of Eqs. (7) and (8) satisfy the existence and uniqueness conditions for the solutions x_s and $u_\alpha = \eta_{k+\alpha}$ for any initial conditions from the domains of definition and continuity of the right-hand sides of Eqs. (7) and (8).

Equations (8) have the standard form of equations of perturbed motion, however, their peculiarity is that the coefficients $q_{s\alpha}$ of the controls u_α have the structural constraints as being partial derivatives of function H, while the number of controls does not exceed the number $n - k$ of ignorable coordinates. Because of this peculiarity we can conceive of cases when a part or even all of the coefficients $q_{s\alpha} \equiv 0$ for solution (5); in the latter case the linear system corresponding to system (8) is uncontrollable by ignorable momenta. If, furthermore, all terms in the functions X_s for solution (5), depending on u_α, turn out to be identically zero, then also the nonlinear system (8) is uncontrollable by ignorable momenta. Sufficient conditions for the controllability of a linear nonstationary system are given by well known theorems (for example, Theorems 20.1 and 20.2 in monograph [2]) the fulfillment of whose conditions also solves the problems of optimal stabilization.

Of special interest is the problem of the optimal stabilization of stable motions (5) by controls on the ignorable coordinates. For Eqs. (8), when all $u_\alpha = 0$, let there exist a positive-definite function $V(t, x_s)$, admitting of an infinitesimal upper bound, whose time derivative by virtue of these equations

$$V^{\cdot} = \frac{\partial V}{\partial t} + \sum_{s=1}^{2k} \frac{\partial V}{\partial x_s} \Big(\sum_{r=1}^{2k} p_{sr} x_r + X_s^{(0)} \Big) \equiv W(t, x_s)$$

is nonpositive. Then motion (5) is stable with respect to x_s $(s = 1,\ldots, 2k)$ for $p_\alpha = p_\alpha(t)$. Let us assume that we wish to make this motion asymptotically stable relative to system (8) with $u_\alpha = u_\alpha^\circ(t, x_s)$ and to minimize the integral

$$I = \int_{t_0}^{\infty} [F(t, x_s) + S(u_\alpha)]dt, \quad S(u_\alpha) = \sum_{i, j=1}^{n-k} \beta_{ij} u_i u_j \tag{10}$$

where $F(t, x_s)$ is some nonnegative function yet to be defined, $S(u_\alpha)$ is a specified positive definite quadratic form with real coefficients $\beta_{ij} = \beta_{ji}$. We obtain the solution of this problem in accordance with Theorem 1.1 in [3].

We set up the expression

$$B[V, t, x_s, u_i] = W(t, x_s) + \sum_{s=1}^{2k} \frac{\partial V}{\partial x_s} \Big[\sum_{i=1}^{n-k} q_{si} u_i + \tag{11}$$

$$X_s(t, x_r, u_i) - X_s^{(0)} \Big] + F(t, x_s) + \sum_{i, j=1}^{n-k} \beta_{ij} u_i u_j$$

which, according to the conditions of the theorem on optimal stabilization in [4], achieves, for $u_\alpha = u_\alpha^\circ$, a minimum equal to zero. The optimal controls u_α° satisfy the equations

$$\frac{\partial B}{\partial u_\alpha} = \sum_{s=1}^{2k} \frac{\partial V}{\partial x_s} \left(q_{s\alpha} + \frac{\partial X_s}{\partial u_\alpha} \right) + 2 \sum_{j=1}^{n-k} \beta_{\alpha j} u_j = 0$$

Since, according to (9),

$$\frac{\partial X_s}{\partial u_\alpha} = \frac{\partial X_s^{(1)}}{\partial u_\alpha} + \frac{\partial X_s^{(2)}}{\partial u_\alpha} = X_{s\alpha} + \sum_{j=1}^{n-k} X_{s\alpha j} u_j$$

where $X_{s\alpha}$ are terms independent of u_j; $X_{s\alpha j}$ are the coefficients of u_j in the expression for $\partial X_s / \partial u_\alpha$, and $X_{s\alpha j} \equiv 0$ for $s = 1, \ldots, k$, we have that these equations are linear algebraic equations

$$\sum_{s=1}^{2k} \frac{\partial V}{\partial x_s} (q_{s\alpha} + X_{s\alpha}) + 2 \sum_{j=1}^{n-k} \beta_{\alpha j}^* u_j = 0$$

Here

$$\beta_{\alpha j}^* = \beta_{\alpha j} + \frac{1}{2} \sum_{s=k+1}^{n} \frac{\partial V}{\partial x_s} X_{s\alpha j}$$

Solving these equations, we find

$$u_\alpha^\circ(t, x_s) = -\frac{1}{2} \sum_{r=1}^{n-k} \frac{\Delta_{r\alpha}}{\Delta} \sum_{i=1}^{2k} \frac{\partial V}{\partial x_i} (q_{ir} + X_{ir}) \tag{12}$$

Here $\Delta_{r\alpha}$ is the cofactor of the element $\beta_{r\alpha}^*$ of the determinant $\Delta = \| \beta_{ij}^* \|$, which, since $\| \beta_{ij} \| > 0$, is positive at least for values of variables x_s, sufficiently small in absolute value. Substituting the values (12) instead of u_α into expression (11) and equating it to zero, we obtain an equation from which we find the function

$$F(t, x_s) = -W(t, x_s) + \sum_{i, j=1}^{n-k} \beta_{ij} u_i^\circ u_j^\circ + \sum_{s=k+1}^{2k} \frac{\partial V}{\partial x_s} X_s^{(2)} \tag{13}$$

The time derivative of function $V(t, x_s)$ by virtue of Eqs. (8) with $u_\alpha = u_\alpha^\circ$ is

$$\frac{dV}{dt} = W(t, x_s) - 2\sum_{i,j=1}^{n-k} \beta_{ij} u_i^\circ u_j^\circ - \sum_{s=k+1}^{2k} \frac{\partial V}{\partial x_s} X_s^{(2)} \tag{14}$$

Thus, we have proven

Theorem 1. If for a system (8), stable with $u_\alpha = 0$, we know a positive definite function $V(t, x_s)$, admitting of an infinitesimal upper bound, then it is the optimal Liapunov function for system (8) optimized by controls (12) with respect to functional (10), (13) under the condition that function (14) is negative definite.

The controls which must here be applied to the system relative to the ignorable coordinates, are determined from Eqs. (7). When the conditions of Theorem 1 are fulfilled the unperturbed motion (5) is asymptotically stable with respect to the variables ξ_j, $\eta_j (j = 1, \ldots, k)$ and the integral (10) is minimized. In such a formulation the quantities $\eta_{k+\alpha} = u_\alpha^\circ (t, x_s)$, defined by Eqs. (12), play the role of optimal controls.

Note 1. If we can take the function $H_1 (t, \xi_j, \eta_j, 0) = H_1^{(0)}$, as the function $V(t, x_s)$ of Theorem 1, expression (13) takes the form

$$F = -\frac{\partial H_1^{(0)}}{\partial t} - \sum_{j=1}^{k} \frac{\partial H_1^{(0)}}{\partial \eta_j} \frac{\partial H_1^{(2)}}{\partial \xi_j} + \sum_{i,j=1}^{n-k} \beta_{ij} u_i^\circ u_j^\circ$$

if the function H_1 is represented in a form analogous to (9). The quantities $X_{s\alpha}$ entering into formula (12) are

$$X_{r\alpha} = \frac{\partial^2 H_1^{(1)}}{\partial \eta_r \partial \eta_\alpha}, \quad X_{k+r,\alpha} = -\frac{\partial^2 H_1^{(1)}}{\partial \xi_r \partial \eta_\alpha} \quad (r = 1, \ldots, k; \alpha = k+1, \ldots, n)$$

In a number of cases we may be interested in a certain modification of the given statement of the problem, when we are required to make the unperturbed motion (5) asymptotically stable not only with respect to the variables $\xi_j, \eta_j (j = 1, \ldots, k)$ but also with respect to the variables η_α $(\alpha = k+1, \ldots, n)$. In this case the role of the controls is played by the quantities $P_\alpha = P_\alpha (t, \xi_j, \eta_i)$. This problem, as the preceding one, can be solved with the aid of Theorem 1.1 in [3]. As one of the possible variants of the solution of the problem, let us examine the case when for motion (5) the function $H_1 (t, \xi_j, \eta_i)$ is positive definite and admits of an infinitesimal upper bound. We take it to be a Liapunov function and we find its derivative relative to the Eqs. (6), (7) of perturbed motion,

$$\frac{dH_1}{dt} = \frac{\partial H_1}{\partial t} + \sum_{\alpha=k+1}^{n} \frac{\partial H_1}{\partial \eta_\alpha} P_\alpha$$

By examining the expression

$$B[H_1, t, \xi_j, \eta_i, P_\alpha] = \frac{\partial H_1}{\partial t} + \sum_{\alpha=k+1}^{n} \frac{\partial H_1}{\partial \eta_\alpha} P_\alpha + F(t, \xi_j, \eta_i) + \sum_{i,j=k+1}^{n} \beta_{ij} P_i P_j.$$

and its partial derivatives with respect to P_α, we find that

$$P_\alpha^\circ = -\frac{1}{2} \sum_{i=k+1}^{n} \frac{\Delta_{i\alpha}}{\Delta} \frac{\partial H_1}{\partial \eta_i}, \quad \Delta = \| \beta_{ij} \| \quad (\alpha = k+1, \ldots, n)$$

$$F(t, \xi_j, \eta_i) = -\frac{\partial H_1}{\partial t} + \sum_{i,j=k+1}^{n} \beta_{ij} P_i^\circ P_j^\circ \tag{15}$$

Consequently, the following theorem is valid.

Theorem 2. If for the unperturbed motion (5) the function $H_1(t, \xi_j, \eta_i)$ is positive definite and admits of an infinitesimal upper bound, it is the optimal Liapunov function for system (6),(7) optimized by controls (15) with respect to the functional

$$I = \int_{t_0}^{\infty} \Big[-\frac{\partial H_1}{\partial t} + \sum_{i,\,j=k+1}^{n} \beta_{ij} P_i^\circ P_j^\circ + \sum_{i,\,j=k+1}^{n} \beta_{ij} P_i P_j \Big] dt \tag{16}$$

under the condition that the function

$$\frac{dH_1}{dt} = \frac{\partial H_1}{\partial t} - 2 \sum_{i,\,j=k+1}^{n} \beta_{ij} P_i^\circ P_j^\circ \tag{17}$$

is negative definite.

We now look at the case when the Hamiltonian function H does not depend explicitly on time and all the nonpotential generalized forces $Q_i = 0$ $(i = 1,\ldots, n)$. For fixed values of $p_\alpha = c_\alpha$ and for specified initial conditions, system (1) admits of the solution

$$q_j = q_{j0},\ q_\alpha = q_{\alpha_0}(t - t_0) + q_{\alpha 0},\ p_i = p_{i0} \tag{18}$$

$$(i = 1, \ldots, n;\ j = 1, \ldots, k;\ \alpha = k + 1, \ldots, n)$$

describing the steady-state motion of the system. In particular, if all $q_{\alpha 0} = p_{i0} = 0$, solution (18) describes the equilibrium of the system. The constants q_{j0}, p_{j0}, $q_{\alpha 0}$ are determined by the equations

$$\frac{\partial H}{\partial p_j} = 0, \qquad \frac{\partial H}{\partial q_j} = 0, \qquad \frac{\partial H}{\partial p_\alpha} = q_{\alpha 0} \tag{19}$$

$$(j = 1, \ldots, k;\ \alpha = k + 1, \ldots, n)$$

The equations of perturbed motion of the reduced system have the form of Eqs. (8) whose right-hand sides for motion (18) do not depend explicitly on time, and, in particular, all coefficients p_{si} and $q_{s\alpha}$ are constants.

As follows from control theory [2], the problem of the optimal control of the linear system

$$\frac{dx_s}{dt} = \sum_{i=1}^{2k} p_{si} x_i + \sum_{\alpha=1}^{n-k} q_{s\alpha} u_\alpha \qquad (s = 1, \ldots, 2k) \tag{20}$$

obtained from system (8) by discarding the nonlinear terms X_s, has a solution if and only if the rank of the matrix

$$K = \{Q,\ PQ, \ldots,\ P^{2k-1}Q\}$$

equals $2k$, where P and Q are the matrices of coefficients p_{si} and $q_{s\alpha}$. Here the control $u_\alpha(t)$, solving the problem of damping the linear system, is determined by the maximum principle [2] (see (17.1)); the solution of the linearized system can be extended also to the original nonlinear problem.

In [5] it was shown that for the optimal stabilization of motion (18) by forces applied to the system relative to the ignorable coordinates, it is sufficient to fulfill the condition rank $K = 2k$ for system (20). For an autonomous system of form (8) this condition can be necessary only if among the roots of the characteristic equation

$$\det \| P - \lambda E_{2k} \| = 0$$

for system (20) there are roots with positive real parts. If the multiplicity of the zero root of this equation is greater than $n - k$, then the indicated stabilization is impossible. The latter assertion is connected with the least possible dimension of the control vector [6].

Let us examine the function $H_1(\xi_j, \eta_i)$ and its time derivative relative to system (6),(7). According to Liapunov's stability theorem the steady-state motion (18) is stable with respect to the position coordinates and to all the momenta if the function $H_1(\xi_j, \eta_i)$ is positive definite in all the variables ξ_j, η_i, while its time derivative is nonpositive definite. Here, if the manifold M, defined by the equation

$$\sum_{\alpha=k+1}^{n} \frac{\partial H_1}{\partial \eta_\alpha} P_\alpha = 0 \tag{21}$$

does not contain the integral motions of the system besides the motion $\xi_j = \eta_i = 0$, then according to the Barbashin-Krasovskii theorem, motion (18) is asymptotically stable with respect to those same variables.

Let us consider, in particular, controls of the form

$$P_\alpha = -\partial H_1 / \partial \eta_\alpha \qquad (\alpha = k+1, \ldots, n) \tag{22}$$

Then

$$\frac{dH_1}{dt} = -\sum_{\alpha=k+1}^{n} \left(\frac{\partial H_1}{\partial \eta_\alpha}\right)^2$$

In this case the manifold M is defined by the equations

$$-P_\alpha = \sum_{i=1}^{k} \left(\frac{\partial^2 H}{\partial q_i \partial p_\alpha} \xi_i + \frac{\partial^2 H}{\partial p_i \partial p_\alpha} \eta_i\right) + \sum_{j=k+1}^{n} \frac{\partial^2 H}{\partial p_\alpha \partial p_j} \eta_j + \ldots = 0 \tag{23}$$
$$(\alpha = k+1, \ldots, n)$$

where the ellipsis denotes the collection of terms of higher than first order in smallness. Since

$$\left\| \frac{\partial^2 H}{\partial p_\alpha \partial p_j} \right\|_{k+1}^{n} > 0$$

Eq. (23) can always be solved with respect to η_α,

$$\eta_\alpha = f_\alpha(\xi_j, \eta_j) \qquad (\alpha = k+1, \ldots, n) \tag{24}$$

Under conditions (23) Eq. (7) has the integrals $\eta_\alpha = \text{const}$, therefore, relations (24) are the first integrals of Eqs. (6). Here we should distinguish whether or not the functions f_α depend on η_j. In the first case relations (24) are integrals linear in η_j. Since linear integrals are possessed only by those dynamic systems which either have ignorable coordinates or can be transformed into systems with ignorable coordinates [7], we have that under the assumption that Eqs. (1) do not admit of any linear integrals whatsoever besides the integrals $p_\alpha = c_\alpha$, there can be no integrals of form (24) in system (6), and manifold (23) does not contain integral motions besides $\xi_j = \eta_i = 0$. In the second case, when the functions f_α do not depend on η_j, such motions are, in general, possible. Thus, we arrive at the following assertion.

Theorem 3. If the function $H_1(\xi_j, \eta_i)$ is positive definite, it is the optimal Liapunov function for the system of autonomous Eqs. (6),(7), optimized by controls (22) with respect to the functional

$$I = \frac{1}{2} \int_{t_0}^{\infty} \left[\sum_{\alpha=k+1}^{n} P_\alpha^2 + \sum_{\alpha=k+1}^{n} \left(\frac{\partial H_1}{\partial \eta_\alpha}\right)^2 \right] dt \tag{25}$$

under the condition that manifold (23) does not contain integral motions of the system besides $\xi_j = \eta_i = 0$.

We note that this result abuts Theorem 2.1 of [8]. Forces of form (22) are dissipative forces $-\xi_\alpha^{\boldsymbol{\cdot}}$ applicable to the system with respect to the appropriate ignorable coordinates.

Note 2. The problem of control and stabilization of the motions of systems with ignorable coordinates includes within itself, as a special case, the problem of control and stabilization of relative motions and of the equilibria of a holonomic system relative to a fixed coordinate system $Ox_1x_2x_3$, performing a known motion. Indeed, the system's motion relative to the fixed axes can be described by Hamiltonian equations of form (3) set up for an absolute motion if as the generalized coordinates q_j we take the independent variables defining the position of the system relative to the moving coordinate system. Here the Hamiltonian function H depends not only on q_j and p_j but also on the projections of the velocity of the origin of the moving coordinate systems and its absolute angular velocity which in equations of form (3) play the role of controls.

Example 1. A heavy point P of mass m is located on a material circle of radius a with center at O, lying in a vertical plane, which can rotate without friction around its vertical diameter. The position of the circle is determined by the angle ψ its plane forms with a certain fixed plane, while the position of the point on the circle is determined by the angle θ between the downward vertical and the radius OP. The moment of inertia of the circle relative to the diameter equals I. By p_1 and p_2 we denote the generalized momenta corresponding to the angles θ and ψ, and we write down the Hamiltonian function

$$H(\theta, p_1, p_2) = \frac{p_1^2}{2ma^2} + \frac{p_2^2}{2(I + ma^2\sin^2\theta)} - mga\cos\theta$$

The equations of form (19) have the roots

$$p_{10} = 0, \qquad \theta_0 = 0, \pi$$

for any constant value $p_2 = c$, as well as the root

$$\theta_0 = \arccos\frac{g}{a\omega^2}, \qquad \omega = \frac{c}{I + ma^2\sin^2\theta_0}, \qquad p_{10} = 0$$

for the values $c \geqslant c_* = I\sqrt{g/a}$. On the (θ, p_2)-plane these solutions are represented by three brances, $\theta = 0$, $\theta = \pi$ and $\theta = \arccos g/a\omega^2$, of the "equilibrium" curve. We see that the steady-state motions are stable relative to θ, p_1, p_2 for points of the first branch for $0 \leqslant c < c_*$ and for points of the third branch for $c > c_*$, and are unstable for points of the first branch for $c > c_*$ and for all points of the second branch.

In this case the function $H_1(\xi, \eta_1, \eta_2)$ has the form

$$H_1(\xi, \eta_1, \eta_2) = \frac{1}{2}\left[\frac{\eta_1^2}{ma^2} + \frac{(c+\eta_2)^2}{I + ma^2\sin^2(\theta_0+\xi)}\right] - \frac{1}{2}\frac{c^2 + 2c\eta_2}{I + ma^2\sin^2\theta_0} - mga\left[\cos(\theta_0+\xi) - \cos\theta_0\right]$$

Since the quantities

$$\frac{\partial^2 H}{\partial\theta\,\partial p_2} = -\frac{ma^2 c\sin 2\theta}{(I + ma^2\sin^2\theta)^2}, \qquad \frac{\partial^2 H}{\partial p_1\,\partial p_2} = 0$$

we have that for all points of the first and second branches the system is uncontrollable in the first approximation by an ignorable momentum, while it is controllable for points of the third branch for which rank $K = 2$. By examining the functions $H_1(\xi, \eta_1, 0)$ and $H_1(\xi, \eta_1, \eta_2)$ and their time derivatives by virtue of Eqs. (6) of perturbed motion

$$\frac{dH_1^{(0)}}{dt} = \frac{1}{2}\frac{(\eta_2^2 + 2c\eta_2)\,\eta_1\sin 2(\theta_0+\xi)}{[I + ma^2\sin^2(\theta_0+\xi)]^2}, \qquad \frac{dH_1}{dt} = \frac{\partial H_1}{\partial\eta_2}P_2$$

according to Theorems 1 and 3 we find that controls of form (12) or (22)

$$\eta_2^\circ = -\frac{c\eta_1 \sin 2(\theta_0 + \xi)}{2\beta[I + ma^2 \sin^2(\theta_0 + \xi)]^2 + \eta_1 \sin 2(\theta_0 + \xi)},$$

$$P_2^\circ = \frac{c}{I + ma^2 \sin^2 \theta_0} - \frac{c + \eta_2}{I + ma^2 \sin^2(\theta_0 + \xi)}$$

stabilize up to asymptotic stability with respect to the variables ξ, η_1 or the variables ξ, η_1, η_2, respectively, the steady-state motions corresponding to the stable points of the first branch. Here they minimize an integral of form (10) or an integral of form (25), since in both cases the manifold M does not contain integral motions besides the unperturbed one. For points of the third branch these manifolds contain the integral motions $\xi = \text{const}$, as a consequence of which the controls indicated ensure only the asymptotic tendency of the perturbed motion to one of the steady-state motions sufficiently close to the unperturbed motion.

Let us now consider the question of stabilizing the relative equilibria of the heavy point on the circle for the case when an external force moment ensuring $\psi^\cdot = \omega = \text{const}$ has been applied to the circle. The Hamiltonian function and the equations of relative motion of the point have the form

$$H(\theta, p_1, \psi^\cdot) = \frac{1}{2}\frac{p_1^2}{ma^2} - \frac{1}{2} ma^2 \psi^{\cdot 2} \sin^2\theta - mga\cos\theta$$

$$\frac{d\theta}{dt} = \frac{p_1}{ma^2}, \qquad \frac{dp_1}{dt} = ma^2\psi^{\cdot 2}\sin\theta\cos\theta - mga\sin\theta$$

We see that the relative equilibria of the point and the nature of their stability are the same as for the steady-state motions examined above. By setting $\theta = \theta_0 + \xi$, $p_1 = \eta$, $\psi = \omega + \zeta$ in the perturbed motion, we consider the function

$$H_1(\xi, \eta) = H(\theta_0 + \xi, \eta, \omega) - H(\theta_0, 0, \omega) = \frac{1}{2}\frac{\eta_1^2}{ma^2} -$$

$$-\frac{1}{2} ma^2 \sin^2(\theta_0 + \xi)\,\omega^2 - mga[\cos(\theta_0 + \xi) - \cos\theta_0] + \frac{1}{2} ma^2\omega^2 \sin^2\theta_0$$

and its time derivative by virtue of the equations of perturbed motion

$$dH_1/dt = {}^1/_2\,\eta\,(\zeta^2 + 2\,\omega\zeta)\sin 2(\theta_0 + \xi)$$

In accordance with Theorem 1 we find that controls of form (12)

$$\zeta^\circ = -\frac{\omega\eta \sin 2(\theta_0 + \xi)}{2\beta + \sin 2(\theta_0 + \xi)\eta}$$

stabilize up to asymptotic stability with respect to the variables ξ, η the relative equilibria corresponding to the stable points of the first branch and minimize an integral of form (10), since manifold M does not contain integral motions besides the unperturbed one. For points of the third branch this manifold M contains the integral motions $\xi = \text{const}$.

Example 2. Let us consider the motion of a heavy gyroscope in a gimbal suspension for the case when the axis of the outer gimbal is vertical. Retaining the notation in [9], we write out the Hamiltonian function

$$H(\theta, p_1, p_2, p_3) = \frac{1}{2}\left[\frac{p_1^2}{A + A_1} + \frac{(p_2 - p_3\cos\theta)^2}{(A + B_1)\sin^2\theta + C_1\cos^2\theta + A_2} + \frac{p_3^2}{C}\right] + Pz_0\cos\theta$$

The equations of the form (19) have solutions for which

$$\theta = \theta_0, \quad p_1 = 0, \quad \psi' = \Omega, \quad r = \omega$$

under the condition that the constants θ_0, Ω, ω satisfy the relation (2.2) in [9]. Setting up the matrix K, we can show that for the solutions $\theta_0 = 0$, π the system in the first approximation is uncontrollable by the ignorable momenta p_2 and p_3, while for the solution $\theta \neq 0$, π is controllable. The steady-state motion, for which $\theta_0 = 0$, is stable if condition (2.8) in [9] is fulfilled. Such a stable motion can be stabilized up to asymptotic stability by forces of form (22) and minimize an integral of form (25).

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Translated by N.H.C.

UDC 531.36+62-50

ON STABILIZATION OF STEADY-STATE MOTIONS OF MECHANICAL SYSTEMS WITH RESPECT TO A PART OF THE VARIABLES

PMM Vol. 36, №6, 1972, pp. 977-985

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(Received May 22, 1972)

We pose the problem of stabilization with respect to position coordinates and velocities of the steady-state motions of holonomic mechanical systems by means of forces acting only on the ignorable coordinates. The problem is reduced to the stabilization of the trivial solution of a certain system of differential equations, in which perturbations of the ignorable momenta are treated as the controls. As an example we examine the asymptotic stabilization of the relative equilibrium positions of a gyrostat satellite in a circular orbit.

1. We consider a holonomic scleronomous mechanical system with n degrees of freedom. Let q_r be the generalized coordinates, $q_r^{\cdot}$, p_r $(r = 1,\ldots, n)$ be the generalized velocities and momenta, T and Π be the kinetic and potential energies, res-